Analogues between a quantum many body problem and the log-gas

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Abstract. The ground state wavefunction of a t-component quantum system in one dimension interacting via the $1/r^2$ potential is interpreted as the Boltzmann probability factor of a t-component log-gas. This allows predictions regarding the asymptotic behaviour of the ground state correlations and density matrix to be made. Using the results of an exact calculation of a two-component log-gas, exact results are obtained for the quantum system in the case $t = 2$.

1. Introduction

The Boltzmann factor for the classical one-component Coulomb gas with the logarithmic potential (log-gas) on the circle is given by

$$\psi_0^2 = A^2 \prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}|^\Gamma \tag{1.1}$$

where

$$\Gamma = q^2 / k_B T \tag{1.2}$$

and A is constant. It was observed by Sutherland (1972) that ψ_0 is also the ground state wavefunction of the N-body Schrödinger equation with

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{g\pi^2}{L^2} \sum_{1 \leq j < k \leq N} \left(\sin \frac{\pi(x_k - x_j)}{L} \right)^{-2}, \tag{1.3}$$

$x_k = L\theta_k / 2\pi$. Note in the limit $L \to \infty$ the potential becomes $V(r) = g/r^2$. The wavefunction is subject to periodic boundary conditions, and is defined to be positive in the region $0 \leq x_1 \leq x_2 \leq \ldots \leq x_N \leq L$, the choice of sign of ψ_0 in other regions depending on the particle type—boson or fermion. The coupling constants Γ and g are related by the equation

$$\Gamma = 1 + \sqrt{1 + 2g}, \qquad g \geq -\tfrac{1}{2}. \tag{1.4}$$

Exact results are available for the partition function of the one-component log-gas at all temperatures, and the n-particle correlations at three particular temperatures. This allows the wavefunction to be normalised for all values of $g \geq -\frac{1}{2}$ (for $g < -\frac{1}{2}$ the phenomenon of 'fall towards the centre' begins), and the n-particle correlations of the ground state wavefunction to be calculated when $g = -\frac{1}{2}$, 0 and 4. Furthermore the

density matrix was explicitly evaluated for fermions when $g = 0$, and for bosons when $g = 4$. Apart from the case $g = 0$, which corresponds to free fermions, these results marked the first explicit calculation of the correlation functions and density matrix of an interacting N-body quantum system.

In this paper we further exploit the analogy between the two problems. In § 2 we collect together known results relevant to both the one-component log-gas and the quantum problem. It is shown how the asymptotic formulae in the quantum problem, for both the two-particle correlations and density matrix for bosons, can be deduced from the log-gas analogue.

In § 3 the t-species analogue of the Hamiltonian (1.3) recently given by Krivnov and Ovchinnikov (1982) is considered. The ground state wavefunction is again interpretable as the Boltzmann factor of a log-gas, this time the log-gas having t components. All analogues between the correlations and density matrix in the quantum problem, and distribution functions in the log-gas, discussed in § 2 for the one-component case, carry over to the multicomponent case. Here the asymptotic formulae thus obtained for the quantum system are new.

Furthermore we can use some recently obtained exact results for the two-component log-gas (Forrester (1984a) to be referred to as I, and Forrester (1984b) to be referred to as II) to provide some exact results for the quantum system when $t = 2$. We calculate the two-particle correlation functions, as well as the density matrix for one of the species in the case of bosons, at a special value of the coupling constant.

We will for the majority of this paper use the language of log-gas statistical mechanics.

2. The single species quantum system

Here we are considering properties of the ground state wavefunction (1.1).

2.1. Pair correlation functions

The ground state pair correlation function, which is defined in terms of $|\psi_0|^2$ and thus independent of the particle type, is identical to the pair correlation function ρ_2^T of the one-component log-gas interacting at the temperature (1.2).

From the general theory of classical Coulomb systems, we know ρ_2^T must obey Jancovici's sum rule (Jancovici 1982, Forrester *et al* 1983). In the present case this says the asymptotic expansion of ρ_2^T must contain the term

$$-1/\Gamma \pi^2 y^2. \tag{2.1}$$

Furthermore, it is expected that above a certain temperature this will be the leading term (Forrester *et al* 1983). This result can be compared to the asymptotic form of the ground state pair correlations of general one-component, one-dimensional quantum fluids given by Haldane (1981)

$$\rho_2^T(y) \sim -\frac{\eta}{4\pi^2 y^2} + \rho^2 \sum_{m=1}^{\infty} A_m (\rho y)^{-m^2\eta} \cos(2\pi m\rho y). \tag{2.2}$$

Here η is known as the correlation exponent, and the A_m are model dependent quantities. The correlation exponent is defined in terms of the compressibility (it should be noted Haldane denotes $\rho \partial p / \partial \rho = \kappa$ rather than the usual κ^{-1}). Furthermore

we point out that formula (2.2) only contains the leading-order non-oscillatory term, the leading-order oscillatory term of period $1/\rho$ etc (higher-order terms of these forms will be present in general). The zero-temperature equation of state for the Hamiltonian (1.1) is given by Sutherland (1971), so κ and thus η can be calculated to give

$$\eta = 4/\Gamma. \tag{2.3}$$

Substituting (2.3) in (2.2) shows Haldane's asymptotic formula is consistent with Jancovici's sum rule. Furthermore for $\Gamma < 2$ we note the term (2.1) represents the leading-order behaviour.

2.2. The density matrix for bosons

We will use the symbol D to denote the density matrix rather than the usual ρ to avoid confusion with the distribution functions. For general one-dimensional, one-component quantum fluids Haldane (1981) gives the asymptotic formula

$$D(y) \sim \frac{\rho}{(\rho y)^{1/\eta}} \left(B_0 + \sum_{m=1}^{\infty} B_m (\rho y)^{-m^2 \eta} \cos 2\pi m \rho y \right) \tag{2.4}$$

where the B's are model dependent quantities, and η is the same as in (2.2). For the Hamiltonian (1.1), when η is given by (2.3), the leading term of the expansion (2.4) can be deduced from a log-gas analogue.

Recall that the density matrix is defined as

$$D(y) = N \left(\prod_{j=1}^{N-1} \int_0^L dx_j \, \psi_0(y) \psi_0(0) \right) \bigg/ Z^2 \tag{2.5}$$

where

$$Z^2 = \prod_{j=1}^{N} \int_0^L dx_j \, \psi_0^2,$$

$$\psi_0(y) = \psi_0(x_1, x_2, \ldots, x_{N-1}, y)$$

and ψ_0 is given by (1.1), with the sign specified in the paragraph subsequent to that equation. Note the density matrix is normalised so that $D(0) = \rho$. We can interpret $D(y)$ as a distribution function in the log-gas by noting

$$\psi_0(y)\psi_0(0) = \phi_0^2 \prod_{l=1}^{N-1} (|\sin(\pi/L)(y - x_i)\| \sin(\pi/L)x_i|)^{\Gamma/2} \tag{2.6}$$

where ϕ_0^2 is the Boltzmann factor for the one-component log-gas of $(N-1)$ particles at the temperature (1.2). If we now interpret the whole right-hand side of (2.6) as a log-gas at the temperature (1.2) the second term in (2.6) represents the Boltzmann factor for the interaction of the $(N-1)$ particles with a particle of charge $q/2$ fixed at y and another particle of charge $q/2$ fixed at the origin. We can now interpret the right-hand side of (2.6) as a Boltzmann factor of $(N-1)$ particles of charge q, and two particles of charge $q/2$, by multiplying by the term $|\sin(\pi/L)y|^{\Gamma/4}$ which is the Boltzmann factor for the charge at the origin interacting with the charge at y.

From the above interpretation of $\psi_0(y)\psi_0(0)$ we see that

$$|\sin(\pi/L)y|^{\Gamma/4} D(y)$$

is a distribution function in the log-gas of $(N-1)$ particles of charge q, with two

particles of charge $q/2$ as the test particles. In the thermodynamic limit we expect $D(y)$ to tend to a well behaved limit whereas

$$|\sin(\pi/L)y|^{\Gamma/4} \sim (\pi y/L)^{\Gamma/4}.$$

Thus if we multiply both sides of the definition of $D(y)$ by

$$(L/\pi)^{\Gamma/4}|\sin(\pi/L)y|^{\Gamma/4},$$

we have a quantity which is well behaved in the thermodynamic limit, and represents the distribution function in the log-gas described above.

This interpretation allows us to predict the large separation behaviour of $D(y)$. A properly normalised distribution function must, in the fluid state with a repulsive potential, equal zero at zero separation of the test particles and as the separation increases from zero assume non-zero values, tending to a positive constant as the separation of the particles tends to infinity. Hence we expect

$$D(y) \sim c/y^{\Gamma/4}, \qquad \text{as } y \to \infty, \qquad c > 0, \tag{2.7}$$

which is precisely the leading-order term of (2.4) given by Haldane (since here η is given by (2.3)).

3. The *t*-species quantum system

In this section the t-species analogue of the Hamiltonian (1.3) is considered. Suppose we have a system consisting of N_k particles of mass m_k, $k = 1, 2, \ldots, t$, with Hamiltonian

$$H = -\sum_{k=1}^{t} \sum_{j=1}^{N_k} \left(\frac{m_1}{m_j}\right) \frac{\partial^2}{\partial(x_j^{(k)})^2} + \left(\frac{\pi}{L}\right)^2 \sum_{k=1}^{t} g_k \sum_{1 \le i < j \le N_k} \sin^{-2} \frac{\pi}{L}(x_j^{(k)} - x_i^{(k)})$$

$$+ \left(\frac{\pi}{L}\right)^2 \sum_{1 \le k < l \le t} g_{kl} \sum_{i=1}^{N_k} \sum_{j=1}^{N_l} \sin^{-2} \frac{\pi}{L}(x_j^{(k)} - x_i^{(l)}). \tag{3.1}$$

The species with mass m_k have coordinates denoted $x^{(k)}$.

Krivnov and Ovchinnikov (1982) show that the ground state (since it is nodeless for finite values of the potential) wavefunction

$$\psi_0^2 = \left(\prod_{k=1}^{t} D^{m_k^2}(x^{(k)}) \prod_{1 \le k < l \le t} D^{m_k m_l}(x^{(k)}, x^{(l)})\right)^{\Gamma/m_1^2} \tag{3.2}$$

where

$$D(x^{(k)}) = \prod_{1 \le i < j \le N_k} \left|\sin \frac{\pi}{L}(x_i^{(k)} - x_j^{(k)})\right|, \tag{3.3}$$

$$D(x^{(k)}, x^{(l)}) = \prod_{i=1}^{N_k} \prod_{j=1}^{N_l} \left|\sin \frac{\pi}{L}(x_i^{(k)} - x_j^{(l)})\right|, \qquad k \ne l \tag{3.4}$$

and

$$\Gamma = m_1^2/k_B T, \tag{3.5}$$

satisfies the Schrödinger equation with H given by (3.1) provided the coupling constants are suitably defined. The definition of Γ (3.5) is consistent with (1.2) if we choose $q = m_1$. Also the sign of the wavefunction is determined as for (1.1). For given values

of the ratios (m_k/m_1), $k = 2, \ldots, t$, the coupling constants g_k and g_{kl} are uniquely determined by the single parameter Γ. We find we require

$$g_k = \Gamma(m_k/m_1)[(\Gamma/2)(m_k/m_1)^2 - 1], \qquad\qquad k = 1, \ldots, t, \qquad (3.6)$$

$$g_{kj} = (\Gamma/2)[(\Gamma/2)(m_k m_j/m_1^2) - 1](m_k + m_j)/m_1, \qquad k \neq j. \qquad (3.7)$$

Notice that Γ is a double valued function of the coupling constants—a feature attributable to the singularity at the origin of the $1/r^2$ potential. This point has been discussed by Sutherland (1971) from whose considerations we conclude the branch of the solution of (3.6) and (3.7) such that Γ is always positive should be chosen. Since there is only one independent coupling constant, g_1 say, this can be achieved by requiring

$$\Gamma = 1 + (1 + 2g_1)^{1/2}, \qquad\qquad g_1 \geqslant -\tfrac{1}{2}. \qquad (3.8)$$

Here our fundamental observation is that ψ_0 is the Boltzmann factor (excluding constant terms) of the t-component log-gas on the circle. The log-gas consists of N_k particles of charge m_k, $k = 1, \ldots, t$, interacting at the temperature given by (3.5). We can now proceed as in § 2 and use this analogy to write down asymptotic formulae for the correlation functions and density matrix of the t-component quantum problem.

3.1. Correlation functions

For t-component classical Coulomb systems Jancovici's sum rule relates to the charge–charge correlation C_2^T, which is defined in terms of the usual one- and two-particle distributions by

$$C_2^T(y - y') = \sum_{\alpha=1}^{t} m_\alpha \rho_\alpha \delta(y - y') + \sum_{\alpha=1}^{t} \sum_{\beta=1}^{t} m_\alpha m_\beta \rho_{\alpha,\beta}^T(y - y'). \qquad (3.9)$$

For the t-component system considered here Jancovici's sum rule says (Jancovici 1982, Forrester *et al* 1983) the asymptotic expansion of $C_2^T(y)$ must contain the term $-k_B T/\pi^2 y^2$, and this will be the leading term for high enough temperatures. Thus the asymptotic expansion of $C_2^T(y)$ for the ground state wavefunction (3.2) must contain the term

$$-m_1^2/\Gamma \pi^2 y^2 \qquad (3.10)$$

and for Γ small enough this will be the leading term.

Information regarding the period of oscillatory terms in the asymptotic expansion of the correlations in the t-component log-gas is also available. If we define the charge density

$$Q = \sum_{j=1}^{t} m_j \rho_j, \qquad (3.11)$$

then by the considerations in II of the mixing properties of the log-gas it follows we would expect the leading-order oscillatory term in the asymptotic expansion of the correlations to have period $1/aQ$. Here a is the smallest positive real number such that am_k is an integer, $k = 1, 2, \ldots, t$. For this to be possible we require the m_k to be rational multiples of the same 'unit' charge.

Also we have predictions for the small distance behaviour of the two-particle correlations in the t-component log-gas. Again by considering the mixing properties, it was concluded in II that the correlation $\rho_{\alpha,\delta}^T$ would have its first and maximum peak

at, to a good approximation, the distance

$$(m_\alpha + m_\delta)/2Q. \tag{3.12}$$

This feature would only be present at sufficiently low temperatures (high temperatures would destroy such ordering), which corresponds to the parameter Γ being sufficiently large. The actual temperature at which this feature persists would be dependent on the charge species.

3.2. The density matrix for bosons

In § 2.2 we deduced that in the one-component case the density matrix for bosons was equivalent to a distribution function in the log-gas. The density matrix $D_{(k)}(y)$ of species k (with mass m_k assumed to be bosons) in the general t-species case admits a similar interpretation. We deduce

$$|y|^{(\Gamma/m_1^2)(m_k^2/4)} D_{(k)}(y) \tag{3.13}$$

is a distribution function in the t-component log-gas with charge density (3.11) and two particles of charge $m_k/2$ as the test particles. Thus arguing as in § 2.2 we deduce the asymptotic behaviour

$$D_{(k)}(y) \sim \frac{c'}{y^{(\Gamma/m_1^2)(m_k^2/4)}}, \qquad \text{as } y \to \infty, \qquad c' > 0. \tag{3.14}$$

The asymptotic formula (3.14) tells us immediately the singularity at the origin in k space of the momentum distribution function, given in terms of the density matrix by

$$n_{(p)}(k) = 2 \int_0^\infty D_{(p)}(y)(\cos 2\pi y k) \, dy. \tag{3.15}$$

From the large y behaviour of $D_{(p)}(y)$ we conclude (see Lighthill (1958))

$$n_{(p)}(k) \underset{k \to 0}{\sim} \begin{cases} A(k)^{\alpha-1}, & \alpha \neq \text{integer}, \\ A'(k)^{\alpha-1} \log k, & \alpha = \text{integer}, \end{cases} \tag{3.16}$$

where $\alpha = (\Gamma/m_1^2)(m_p^2/4)$ and A, A' are constants. Since $\alpha > 0$ there is no macroscopic occupation of the zero momentum state.

As with the ground state correlations, we can also predict the small separation behaviour of the density matrix. Applying the mixing considerations of the log-gas to the test particles in the distribution function (3.13), we conclude the distribution function (3.13) will to a good approximation have its first and maximum peak at

$$m_k/2Q. \tag{3.17}$$

The log-gas distribution function analogue can also be used to predict the period of the leading-order oscillatory term in the asymptotic expansion of $D_{(k)}(y)$. Again the mixing considerations of II tell us the leading-order oscillatory term will have period $1/aQ$ (with a and the restrictions on the m_k defined in § 3.1).

3.3. Density matrix for fermions

We have no log-gas analogue for fermions. The integrand of the integral defining the density matrix can assume negative values and is thus not interpretable as a Boltzmann factor.

4. Exact results for a particular two-species system

The two-species log-gas, consisting of N_1 particles of charge $+q$ and N_2 particles of charge $+2q$ has been solved (I and II) at the special value of the coupling constant $q^2/k_B T = 1$. From the results of § 3 above, we can thus normalise the ground state wavefunction of the Hamiltonian (3.1), in the case $t = 2$, $m_2/m_1 = 2$, $g_1 = -\frac{1}{2}$, $g_{12} = 0$ and $g_2 = 2$, and also calculate the ground state correlations by using the log-gas analogue directly. We will also find it possible to calculate the density matrix in the case of species 2 when this species are bosons.

Note that $g_{12} = 0$ yet the ground state wavefunction (3.2) has factors corresponding to interaction between the species. The interpretation of this feature is that the two species have an infinite hard core, and are thus impenetrable.

4.1. Normalisation of the wavefunction

In the case of the Hamiltonian (3.1) specified above, the (un-normalised) ground state wavefunction is given by (3.2) with $(m_2/m_1) = 2$ and $\Gamma = 1$. Normalisation is achieved by requiring

$$A^2 \prod_{k=1}^{2} \prod_{l=1}^{N_l} \int_0^L dx_l^{(k)} \, \psi_0^2 = 1 \tag{4.1}$$

where A is the normalisation constant. From the calculation of the partition function for the exactly solvable two-species log-gas specified above (I equation (2.16)) we have

$$A^{-2} = \left(\frac{L}{2\pi}\right)^{N_1+N_2} \frac{N_1! \, N_2! (N_2 + N_1/2)!}{(2N_2 + N_1)!} (16\pi)^{N_1/2+N_2} \sum_c \prod_{l=1}^{N_2} (c(l) - \tfrac{1}{2})^2 \tag{4.2}$$

where the sum is over all combinations of $\{1, 2, \ldots, N_2 + N_1/2\}$ taken N_2 at a time.

4.2. The ground state correlation functions

From § 3.1 above, we know the ground state correlations of the quantum system, and its log-gas analogue, are the same. Thus in the present case, we refer to II (§ 3), where the correlations of the log-gas are given to obtain

$$\rho_{m_1,m_1}^T(y) = -(P^2/\nu^4)(I_1 I_3 + I_2^2) + (\pi/2\nu^2)I_3, \tag{4.3}$$

$$\rho_{m_1,m_2}^T(y) = -(P^2/2\nu^2)(I_2 I_4 + I_3^2), \tag{4.4}$$

$$\rho_{m_2,m_2}^T(y) = -(P^2/4)(I_3 I_5 + I_4^2), \tag{4.5}$$

where

$$I_1 = \int_0^1 dt \frac{\sin \pi y P t}{t(t^2 + 1/\nu^2)}, \qquad I_2 = \int_0^1 dt \frac{\cos \pi y P t}{t^2 + 1/\nu^2},$$

$$I_3 = \int_0^1 dt \frac{t \sin \pi y P t}{t^2 + 1/\nu^2}, \qquad I_4 = \int_0^1 dt \frac{t^2 \cos \pi y P t}{t^2 + 1/\nu^2},$$

$$I_5 = \int_0^1 dt \frac{t^3 \sin \pi y P t}{t^2 + 1/\nu^2}.$$

Here ν is the unique positive solution of the equation

$$N_1/(N_1 + 2N_2) = (\tan^{-1} \nu)/\nu \tag{4.6}$$

and $P = Q/m_1$ where Q, the charge density, is given by (3.11) with $t = 2$.

It is verified in I that the charge–charge correlation satisfies Jancovici's sum rule. All oscillatory terms (and thus the leading-order one) have period $1(\rho_1 + 2\rho_2)$. This is in agreement with our expectations stated in § 3.1 above, since $a = m_1$. Also in II it was shown, by plotting the correlations for small y, that to a good approximation the first and maximum peak occurs at a distance given by (3.12).

4.3. Calculation of the density matrix for species 2 in the case of bosons

In this case, the log-gas analogue given in § 3.2 above is a distribution function within the two-species log-gas of N_1 particles of charge m_1 and $(N_2 - 1)$ particles of charge m_2, with two particles of charge $\frac{1}{2}m_2$ as the test particles. However here $\frac{1}{2}m_2 = m_1$ so this is (apart from a normalisation factor) the distribution function $\rho_{m_1,m_1}(y)$. Taking into account the normalisations of $D_{(2)}(y)$ and $\rho_{m_1,m_1}(y)$ we readily find

$$D_{(2)}(y) = \frac{\nu^2}{\pi^2 P^2}\left(\frac{1}{|y|}\right)(\rho_{+1}^2 + \rho_{m_1,m_1}^{\mathrm{T}}(y)) \tag{4.7}$$

where $\rho_{m_1,m_1}^{\mathrm{T}}(y)$ is given by (4.3).

Since $m_2/m_1 = 2$ and $\Gamma = 1$ we see immediately the expected asymptotic behaviour (3.14) is obeyed. Furthermore our expectations regarding the location of the first peak, and the period of oscillatory terms in the asymptotic expansion, are the same as for $\rho_{m_1,m_1}(y)$ and thus hold. Note in the limit $\rho_1 \to 0$ (and thus $\nu \to \infty$) we regain a one-component system of particles of mass $2m_1$ with $\Gamma = 1$, or what is the same system, particles of mass m_1 with $\Gamma = 4$. This is the system for which Sutherland (1972) evaluated the density matrix. After some simple analysis we regain his result

$$D(y) = \frac{1}{2\pi y} \int_0^{2\pi\rho y} \frac{\sin z}{z} \, dz. \tag{4.8}$$

From the expression for the density matrix (4.7) we can evaluate the momentum distribution function $n_{(2)}(k)$ defined by (3.15). For purposes of taking the cosine transform it is useful to use the representation of $\rho_{m_1,m_1}(y)$ given in I to write (4.7) in the form

$$D_{(2)}(y) = \frac{1}{2\pi y}\int_0^1 dt \frac{t \sin \pi P y t}{t^2 + 1/\nu^2} - \frac{1}{2\pi^2\nu^2}\left(\frac{1}{|y|}\right)$$
$$\times \int_0^1 \int_{-1}^1 dt \, ds \frac{(t-s)^2 \sin^2 \pi y P(t+s)/2}{st(1/\nu^2 + t^2)(1/\nu^2 + s^2)}. \tag{4.9}$$

The integrations over y can now be performed by the use of tabulated integrals (Gradshteyn and Ryzhik 1965) to obtain

$$n_{(2)}(k) = \frac{-2\nu^2\rho_1^2}{\pi^2 P^2} \log(2k/P) - \frac{1}{4\pi^2\nu^2}\int_0^1 \int_{-1}^1 dt \, ds \frac{(t-s)^2 \log|(t+s)^2 - (2k/P)^2|}{st(1/\nu^2 + t^2)(1/\nu^2 + s^2)}$$
$$+ \begin{cases} \frac{1}{4}(\log(1 + 1/\nu^2) - \log(1/\nu^2 + (2k/P)^2)), & (2k/P) < 1, \\ 0, & (2k/P) \geq 1. \end{cases} \tag{4.10}$$

From (4.10) we obtain the asymptotic behaviour

$$n_{(2)}(k) \underset{k \to 0}{\sim} -(2\nu^2\rho_1^2/\pi^2 P^2)\log(2k/P) \tag{4.11}$$

and

$$n_{(2)}(k) \underset{k \to \infty}{\sim} \left(\frac{\rho_1\nu^2}{3(\rho_1+2\rho_2)} - \frac{2\rho_1\rho_2}{(\rho_1+2\rho_2)^2} - \frac{4\rho_2^2}{(\rho_1+2\rho_2)^2}\right)\frac{P^4}{(\nu\pi)^2(2k)^4}. \tag{4.12}$$

Equation (4.11) is in agreement with the expected behaviour (3.16) (since here $m_2/m_1 = 2$ and $\Gamma = 1$ so that $\alpha = 1$).

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